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Identifiability Of Linear Systems In Physical Coordinates

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Abstract

Identifiability of linear, time-invariant systems in physical coordinates is discussed in this paper. It is shown that identification of the system matrix in physical coordinates can be accomplished by determining a transformation matrix that relates the physical locations of actuators and sensors to the test-data-derived input and output matrices. For systems with symmetric matrices, the solution of a constrained optimization problem is used to characterize all the possible solutions of the transformation matrix. Conditions for the existence of a unique transformation matrix are established easily from the explicit form of the solutions. For systems with limited inputs and outputs, the question about which part of the system can be uniquely identified is also answered. A simple mass-spring system is used to verify the conclusions of this study.

Introduction

Modal updating has gained some research interest largely because the finite-element models of structural systems are usually not accurate enough for dynamic simulation and control design purposes. Most modal updating techniques were developed with the goal of using experimental data to improve the accuracy of finite-element models. Attempts include using constrained parameter optimization, nonlinear dynamic programming, and complicated iterative schemes to update the finite-element model so that it will "fit" a

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given large set of test data. Similar concepts also have been used to develop damage detection techniques. The goal aims for "updating" a mathematical model of the undamaged structure by using experimental data of the damaged structure and then, using the results to detect the locations of damaged structure elements. However, uniqueness of the solution has not been seriously investigated. If the uniqueness question is not answered, one will never be able to settle the dispute about whether model updating and damage detection techniques will really work in practical situations or not.

This paper discusses identifiability of linear, time invariant systems with a limited number of inputs and outputs. The main conclusion is that if there are not enough actuators and sensors, the system model in physical coordinates can not be uniquely determined. Consequently, all existing damage detection techniques, no matter how sophisticated, can never exactly predict the locations of damaged parts of a structure. It is also pointed out in this paper that by knowing the physical locations of the actuators and sensors, it is possible to exactly identify a part of the physical-coordinates model. Therefore, if one's purpose is to update a physical model of the structure, it is important to place actuators and sensors at the right locations.

The organization of this paper is as follows. First, the identification problem considered here is described by a statement of the problem. It shows that the identification of system matrix can be accomplished by determining a transformation matrix. Then, identifiability of systems with unsymmetric matrices is discussed. After that, a constrained optimization problem is solved with the results to be used in the discussion of identifiability of systems with symmetric matrices. The conclusions of this study are basically the same as those of Ref. [1]. However, here we explicitly solve for all the possible solutions of the transformation matrix. Therefore, conditions for the existence of a unique transformation matrix can be established very easily from the form of the solution. The question about which part of the system is exactly identified is also answered. A simple numerical example is included.

Statement of Problem

The system identification problem considered here is described as follows: Given an input-output transfer function matrix $G(s) \in R^{m \times l}$ (derived from test data) and an input matrix $B \in R^{n \times l}$ and an output matrix $C \in R^{m \times n}$ (derived from physical locations of actuators and sensors), determine a matrix $A \in R^{n \times n}$ such that

$$C(sI - A)^{-1}B = G(s) \tag{1}$$

We are interested in the uniqueness of the solution. If A is uniquely determined, then it represents the system matrix in physical coordinates. For model updating and structural redesign purposes, it is important to assure that the system matrix identified is indeed in physical coordinates.

For a given transfer function, there is no unique state-space representation. Existing system realization techniques can find at least one state-space representation $(\bar{A}, \bar{B}, \bar{C})$ that also satisfies $G(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B}$. Since the two triples (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ describe the same system, they are related by a similarity transformation

$$\begin{aligned} \bar{A} &= P^{-1}AP \\ \bar{B} &= P^{-1}B \\ \bar{C} &= CP \end{aligned} \tag{2}$$

$P \in R^{n \times n}$ is called the *transformation matrix.* Therefore, identification of the A matrix in physical coordinates can be accomplished by determining the transformation matrix P, which relates the physical-coordinates model (A, B, C) to a given test-data-derived model $(\bar{A}, \bar{B}, \bar{C})$.

In the following sections, we will show that the transformation matrix is generally not unique if there is not enough sensors and actuators. Identification of first-order systems with unsymmetric system matrices will be discussed first. Then, a constrained optimization problem will be solved along with uniqueness of the solution studied. The results are used to discuss identifiability of symmetric system matrices and mass and stiffness matrices of structural dynamics systems.

Identification of Unsymmetric System matrix

For general linear systems with unsymmetric system matrices, the transformation matrix can be determined from solving Eqs. (2b) and (2c), which are rewritten as

$$\begin{aligned} B &= P\bar{B} \\ \bar{C} &= CP \end{aligned} \tag{3}$$

Let the singular value decompositions (SVD) of $\bar{B}$ and C be

$$\bar{B} = U_B \Sigma_B V_{\bar{B}}^T \quad , \quad C = U_C \Sigma_C V_C^T \tag{4}$$

Then, by premultiplying and postmultiplying Eqs. (3) by appropriate matrices and by using the orthogonality property of singular value decompositions, Eq. (3) can be rewritten as

$$\begin{aligned} V_C^T B V_B &= V_C^T P U_B \Sigma_B \\ U_C^T \bar{C} U_B &= \Sigma_C V_C^T P U_{\bar{B}} \end{aligned} \tag{5}$$

Define a new matrix $\bar{P} \equiv V_C^T P U_{\bar{B}}$, then the above equations become

$$\begin{aligned} V_C^T B V_B &= \bar{P} \Sigma_{\bar{B}} \\ U_C^T \bar{C} U_{\bar{B}} &= \Sigma_C \bar{P} \end{aligned} \tag{6}$$

Therefore, determination of P can be accomplished from determining $\bar{P}$.

Let $\bar{P}$ matrix be partitioned into

$$\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix} \qquad \begin{matrix} \bar{P}_{11} \in R^{m \times l} & , & \bar{P}_{12} \in R^{m \times (n-l)} \\ \bar{P}_{21} \in R^{(n-m) \times l} & , & \bar{P}_{22} \in R^{(n-m) \times (n-l)} \end{matrix} \tag{7}$$

Since $\bar{B} \in R^{n \times l}$ and is a full-rank matrix, $\bar{B}$ has l nonzero singular values. That is, $\Sigma_B = \begin{bmatrix} [\,(\sigma_i)_B\,] \\ 0 \end{bmatrix}$, where $[\,(\sigma_i)_{\bar{B}}\,]$ is an $l \times l$ diagonal matrix with nonzero singular values on the diagonal. Therefore, Eq. (6a) uniquely determines the first l columns of $\bar{P}$ matrix. That is,

$$\begin{bmatrix} \bar{P}_{11} \\ \bar{P}_{21} \end{bmatrix} = V_C^T B V_B [\,(\sigma_i)_{\bar{B}}\,]^{-1} \tag{8}$$

Similarly, because C matrix has m nonzero singular values, Eq. (6b) uniquely determines the first m rows of $\bar{P}$ matrix. That is

$$\begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \end{bmatrix} = [\,(\sigma_i)_C\,]^{-1} U_C^T \bar{C} U_{\bar{B}} \tag{9}$$

There is only one partition, that is $\bar{P}_{22}$, that does not appear in the solutions, Eq. (8) and Eq. (9). If $l \geq n$ and rank$(\bar{B}) = n$, which implies the case that the number of actuators is greater than or equal to the number of states, then $\Sigma_{\bar{B}}$ is a full rank matrix, and hence, Eq. (6a) uniquely determines $\bar{P}$. Similarly, if $m \geq n$ and rank$(C) = n$, then Eq. (6b) uniquely determines $\bar{P}$. If both l and m are less than n, then the solution of $\bar{P}$ is non-unique, because $\bar{P}_{22}$ is undetermined.

Therefore, the necessary and sufficient condition for the transformation matrix P to be uniquely determined is that either rank$(\bar{B})$=n or rank(C)=n. In other words, there must be at least as many sensors or as many actuators as the order of the system so that the A matrix can be uniquely identified. Also note that $\bar{P}_{11}$ is determined twice in Eqs. (8) and (9). The $\bar{P}_{11}$ determined from Eqs. (8) and (9) can be expressed, respectively, by

$$\bar{P}_{11} = V_{C1}^T B V_{\bar{B}} [\,(\sigma_i)_{\bar{B}}\,]^{-1} \quad , \quad \bar{P}_{11} = [\,(\sigma_i)_C\,]^{-1} U_C^T \bar{C} U_{\bar{B}1} \tag{10}$$

where V_{C1}^T is the first m rows of V_C^T and $U_{\bar{B}1}$ is the first l columns of $U_{\bar{B}}$. The above two solutions of $\bar{P}_{11}$ must be equal for consistency. In fact,

$$V_{C1}^T B V_{\bar{B}} [\,(\sigma_i)_{\bar{B}}\,]^{-1} = [\,(\sigma_i)_C\,]^{-1} U_C^T \bar{C} U_{\bar{B}1} \quad \longrightarrow \quad [\,(\sigma_i)_C\,] V_{C1}^T B V_{\bar{B}} = U_C^T \bar{C} U_{\bar{B}1} [\,(\sigma_i)_{\bar{B}}\,]$$

$$\longrightarrow \quad U_C [\,(\sigma_i)_C\,] V_{C1}^T B V_{\bar{B}} V_{\bar{B}}^T = U_C U_C^T \bar{C} U_{\bar{B}1} [\,(\sigma_i)_{\bar{B}}\,] V_{\bar{B}}^T \quad \longrightarrow \quad CB = \bar{C}\bar{B}$$

The condition that $CB = \bar{C}\bar{B}$ must be satisfied if (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ indeed represent the same system. Therefore, if the two solution of $\bar{P}_{11}$ in Eq. (10) are not consistent, then the given test-data-derived model $(\bar{A}, \bar{B}, \bar{C})$ is a wrong model.

Before we move on to discuss identification of systems with symmetric matrix, let us study the solution of a constrained optimization problem. Non-uniqueness of the solution of this optimization problem will prove to be useful in the discussion of identifiability of symmetric system matrices.

A Constrained Optimization Problem

The constrained optimization problem considered is: Given two matrices Q and R, both $\in R^{p\times n}$ and of ranks $\leq n$ (p can be $>$ or $< n$), and a positive-definite symmetric matrix $M \in R^{n\times n}$, find all $P \in R^{n\times n}$ that

$$
\begin{aligned}
&\text{minimize} \;\; J = \| Q - RP \|_F^2 \\
&\text{subject to} \;\; P^T M P = I_n
\end{aligned}
\tag{11}
$$

This optimization problem explores the possibility of "rotating" R into Q through a transformation matrix P which is orthonormal with respect to M. This optimization problem is also closely related to the following constrained algebraic problem:

$$
\begin{aligned}
Q - RP &= 0 \\
P^T M P &= I_n
\end{aligned}
\tag{12}
$$

We will show later that identification of linear systems with symmetric matrices can be formulated as solving the above constrained algebraic equations. Because $\| Q - RP \|_F^2 \geq 0$ for all P and equality holds only when $Q - RP = 0$, the smallest possible minimum value of J is 0. Therefore, every P that satisfies Eq. (12) is also a solution of the optimization problem in Eq. (11).

The derivation for a solution of the constrained optimization problem in Eq. (11) follows the derivation in Ref. [2] for a similar constrained optimization problem. First, J can be rewritten as

$$
J = \text{trace}(Q^T Q) + \text{trace}(P^T R^T R P) - 2\text{trace}(P^T R^T Q)
\tag{13}
$$

Note that if P satisfies the constrained condition $P^T M P = I_n$, then $P^T M^{\frac{1}{2}} M^{\frac{1}{2}} P = I_n$ and so, $M^{\frac{1}{2}} P P^T M^{\frac{1}{2}} = I_n$. The second term in the Eq. (13) can be converted to a form that is independent of P.

$$
\begin{aligned}
\text{trace}(P^T R^T R P) &= \text{trace}(P^T M^{\frac{1}{2}} M^{-\frac{1}{2}} R^T R M^{-\frac{1}{2}} M^{\frac{1}{2}} P) \\
&= \text{trace}(M^{-\frac{1}{2}} R^T R M^{-\frac{1}{2}} M^{\frac{1}{2}} P P^T M^{\frac{1}{2}}) \\
&= \text{trace}(M^{-\frac{1}{2}} R^T R M^{-\frac{1}{2}})
\end{aligned}
$$

Therefore, from Eq. (13) it is seen that minimizing J is equivalent to maximizing $\text{trace}(P^T R^T Q)$. The maximizing P can be obtained by using the singular value decomposition technique. Let the SVD of $M^{-\frac{1}{2}} R^T Q$ be

$$
M^{-\frac{1}{2}} R^T Q = U \Sigma V^T
$$

Since both rank(Q) and rank(R) are $\leq n$, the rank of R^TQ is also $\leq n$. Assume R^TQ has r nonzero singular values ($r \leq n$ and r is not necessarily equal to p) and arrange the decomposition such that

$$\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots) \quad , \quad \sigma_i > 0$$

Define an S matrix by

$$S^T = V^T P^T M^{\frac{1}{2}} U$$

The constrained condition $P^T M P = I_n$ implies that S is orthogonal, i.e., $S^T S = I_n$. Then,

$$\text{trace}(P^T R^T Q) = \text{trace}(P^T M^{\frac{1}{2}} U \Sigma V^T) = \text{trace}(S^T \Sigma) = \sum_{i=1}^{r} s_{ii}\sigma_i$$

in which s_{ii}, $i = 1, \ldots, p$, are diagonal elements of S. Because S is an orthogonal matrix, $s_{ii} \leq 1$. Therefore, the maximum value of trace($P^T R^T Q$) is $\sum_{i=1}^{r} \sigma_i$, which is attained by setting

$$S = \begin{bmatrix} I_r & 0 \\ 0 & S_b \end{bmatrix} \quad \text{with} \quad S_b^T S_b = I_{n-r} \tag{14}$$

The above derivation shows that the solution of the constrained optimization problem in Eq. (11) can be expressed by

$$P = M^{-\frac{1}{2}} U S V^T \tag{15}$$

where S is given by Eq. (14) and U and V are SVD of $M^{-\frac{1}{2}} R^T Q$. Obviously, if $r < n$, then the solution is non-unique because every S matrix in the form of Eq. (14) yields a minimum. Therefore, Eq. (15) represents a "solution group", which includes all the possible solutions. Although every P in the solution group minimizes J, there is only one minimum value of J, that is

$$J_{\min} = \text{trace}(Q^T Q) + \text{trace}(M^{-\frac{1}{2}} R^T R M^{\frac{1}{2}}) - 2\sum_{i=1}^{r} \sigma_i$$

The solution group given in Eq. (15) is also the solution of the constrained algebraic problem in Eq. (12). This statement is deduced from the following arguments. As mentioned previously, if there exists a P that satisfies Eq. (12), then it must be a solution of the optimization problem in Eq. (11) because $J = \| Q - RP \|_F^2 \geq 0$ for all P with equality holds only when $Q - RP = 0$. All the solutions of the optimization problem are included by Eq. (15). If the constrained algebraic problem has any solution at all, the solution must also be included by Eq. (15). Since every solution in the solution group produces the same minimum, if one solution yields $J_{\min} = 0$, then every solution in the solution group yields $J_{\min} = 0$. In other words, the constrained algebraic problem in Eq. (12) either has no solution or has the solution given by Eq. (15). When the constrained algebraic problem has solution, the solution is unique if and only if rank(R^TQ)=n, which implies rank(Q)=rank(R)=n. In the following sections, we will use this conclusion to discuss identifiability of systems with symmetric matrices.

Identification of Symmetric System Matrix

A linear system with symmetric system matrix has real eigenvalues and a set of orthogonal, real eigenvectors. By using eigenvalue decomposition, any given test-data-derived model $(\bar{A}, \bar{B}, \bar{C})$ can be transformed into the modal coordinates representation $(\Lambda, \bar{B}, \bar{C})$, where $\Lambda \in R^{n \times n}$ is a diagonal matrix with system eigenvalues (or poles) on the diagonal. Therefore, identification of A matrix in physical coordinates can be accomplished from finding the transformation matrix that relates the modal-coordinates model and the physical-coordinates model. The modal-coordinates model and the physical-coordinates model are related by the following congruence transformation

$$\Lambda = P^T A P \quad , \quad \bar{B} = P^T B \quad , \quad \bar{C} = CP \tag{16}$$

where P is called the *modal matrix* and satisfies $P^T P = I_n$.

From the last two equations in Eq. (16), it is seen that the modal matrix P can be determined from solving the following constrained algebraic problem

$$\begin{aligned} \begin{bmatrix} \bar{B}^T \\ \bar{C} \end{bmatrix} &= \begin{bmatrix} B^T \\ C \end{bmatrix} P \\ P^T P &= I_n \end{aligned} \tag{17}$$

If P is uniquely determined, then the identification of A is accomplished by setting $A = P\Lambda P^T$. From the results derived in the previous section, P is unique if and only if both $\begin{bmatrix} \bar{B}^T \\ \bar{C} \end{bmatrix}$ and $\begin{bmatrix} B^T \\ C \end{bmatrix}$ have rank equal to n. In other words, if the number of actuators plus the number of sensors is less than the number of the states, then the system matrix in physical coordinates is not identifiable.

Identification of Undamped Structural Dynamics Systems

Now, we will discuss identifiability of undamped structural dynamics systems with symmetric mass and damping matrices. Assume the testing environment is perfectly noise free. Then, the test data of an undamped structural dynamics system can be realized by the following modal model

$$\begin{aligned} &\ddot{\eta} + \Omega^2 \eta = \bar{F} u && \eta \in R^n, \ u \in R^l \\ &y = \bar{C}_d \eta + \bar{C}_v \dot{\eta} + \bar{C}_a \ddot{\eta} && y \in R^m \end{aligned} \tag{18}$$

where $\Omega^2 = \mathrm{diag}\,[\omega_i^2]$ and ω_i are system's natural frequencies. In physical coordinates, the system equation is described by

$$\begin{aligned} &M\ddot{x} + Kx = Fu && x \in R^n, \ u \in R^l \\ &y = C_d x + C_v \dot{x} + C_a \ddot{x} && y \in R^m \end{aligned} \tag{19}$$

It is assumed that physical location of actuators and sensors are given. That is, matrices F, C_d, C_v, and C_a are known. We are interested in identifying M and K in physical coordinates from using the given modal model in Eq. (18) and the given physical locations of actuators and sensors.

These modal-coordinates model and the physical-coordinates model are related by a modal matrix P such that

$$\begin{aligned} &P^T M P = I_n \ , \quad P^T K P = \Omega^2 \\ &\bar{F} = P^T F \ , \quad \bar{C}_d = C_d P \ , \quad \bar{C}_v = C_v P \ , \quad \bar{C}_a = C_a P \end{aligned}$$

First, consider the case when the mass matrix M is given. Then, the transformation matrix P can be determined from solving the following constrained algebraic problem:

$$\begin{bmatrix} \bar{F}^T \\ \bar{C}_d \\ \bar{C}_v \\ \bar{C}_a \end{bmatrix} = \begin{bmatrix} F^T \\ C_d \\ C_v \\ C_a \end{bmatrix} P \tag{20}$$

$$P^T M P = I_n$$

If P is uniquely determined, then the physical stiffness matrix is determined from setting $K = P^{-T}\Omega^2 P^{-1}$. Previous discussion shows that the total number of actuators and sensors must be at least equal to the number of degrees-of-freedom of the system in order for the modal matrix to be uniquely determined.

For the case that both M and K are unknown, the only equation available for determining P is

$$\begin{bmatrix} \bar{F}^T \\ \bar{C}_d \\ \bar{C}_v \\ \bar{C}_a \end{bmatrix} = \begin{bmatrix} F^T \\ C_d \\ C_v \\ C_a \end{bmatrix} P \tag{21}$$

which is an unconstrained algebraic equation. After P is obtained, M and K can be calculated by

$$M = P^{-T} P^{-1} \quad , \quad K = P^{-T}\Omega^2 P^{-1}$$

In order for a unique P to exist, the matrix formed by the input and output matrices must be consistent and must have rank n. Therefore, the knowledge about M does not relieve the number requirement of actuators and sensors.

Which Part of the System is Uniquely and Exactly Identified?

We have shown that if the number of actuators and sensors are not enough, then it is never possible to identify the physical system matrices uniquely. Nevertheless, it is still of interest to know if it's possible to uniquely identify certain part of the system matrices by using the knowledge about physical locations of actuators and sensors.

First, consider the case with unsymmetric system matrices. The physical-coordinates model (A, B, C) and the test-data-derived model $(\bar{A}, \bar{B}, \bar{C})$ are related by a transformation matrix P as shown in Eq. (2). The transformation matrix P is determined from solving Eqs. (2b) and (2c). Previous results shows that unless either rank(B)=n or rank(C)=n, P can not be uniquely determined. Now, assume there is only one sensor and it is located at the i-th physical coordinate. In this case, $C = e_i$, where e_i the i-th row of the identity matrix I_n. Then, the relation $\bar{C} = CP = e_i P$ uniquely determines the i-th row of P matrix. Similarly, if we assume that there is only one actuator and it is located at the j-th physical coordinate, then the relation $\bar{B} = P^{-1}B = P^{-1}e_j^T$ uniquely determines the j-th column of P^{-1}. With the i-th row of P and the j-th column of P^{-1} being uniquely determined, the (i, j)-th element of A matrix is also uniquely determined, since $A = P\bar{A}P^{-1}$.

For systems with symmetric matrices, the test data acquired in a noise-free environment can be realized by a modal model $(\Lambda, \bar{B}, \bar{C})$. The physical-coordinates model (A, B, C) and the modal-coordinates model $(\Lambda, \bar{B}, \bar{C})$ are related by a modal matrix P as shown in Eq. (16). Because $P^{-1} = P^T$, either a sensor or an actuator suffices to determine one row of the modal matrix uniquely. A pair of noncolocated sensor and actuator can identify two rows of the modal matrix, and therefore, four elements of the system matrix.

In summary, from knowing the physical locations of actuators and sensors, it is possible to identify part of the system matrix in physical coordinates. However, this conclusion is based on the assumption that the testing environment is noise free. In practical situations, the test data are inevitably contaminated by noises and small errors. Sensitivity of the identification results to noises is another topic worthy of studying.

A Simple Example

As an example we will consider the simple mass-spring system shown in Fig. 1. There is a displacement sensor and a force actuator located at DOF2 and DOF3, denoted by s and a, respectively. The modal model of this system is given by

$$\ddot{\eta} + \begin{bmatrix} 0.2708 & 0 & 0 \\ 0 & 2.1315 & 0 \\ 0 & 0 & 5.8477 \end{bmatrix} \eta = \begin{Bmatrix} 0.4319 \\ -0.5524 \\ 0.0917 \end{Bmatrix} u$$

$$y = \begin{bmatrix} 0.3539 & 0.2325 & -0.2659 \end{bmatrix} \eta$$

Assume that the mass matrix

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is given. And, we want to determine the physical stiffness matrix K.

The modal matrix P is determined by solving the following constrained algebraic problem:

$$\begin{bmatrix} 0.4319 & -0.5524 & 0.0917 \\ 0.3539 & 0.2325 & -0.2659 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} P \quad , \quad P^T M P = I_3$$

Or,

$$Q = RP \quad , \quad P^T M P = I_3$$

Singular value decompositions of $M^{-\frac{1}{2}} R^T Q (= USV)$ shows that there are two nonzero singular values, while the system has three degrees-of-freedom. Therefore, the solution is not unique. The solution group is given by $P = M^{-\frac{1}{2}} USV^T$ for all S that satisfy

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & S_b \end{bmatrix} \quad , \quad S_b^2 = 1$$

There are two solutions: $S_b = 1$ and $S_b = -1$. Therefore, there are two modal matrix that are solutions of the constrained algebraic problem. They are:

$$P_1 = \begin{bmatrix} 0.2510 & 0.2945 & 0.5918 \\ 0.3539 & 0.2325 & -0.2659 \\ 0.4319 & -0.5524 & 0.0917 \end{bmatrix} \quad , \quad P_2 = \begin{bmatrix} -0.2510 & -0.2945 & -0.5918 \\ 0.3539 & 0.2325 & -0.2659 \\ 0.4319 & -0.5524 & 0.0917 \end{bmatrix}$$

Using these two modal matrices to determine the stiffness matrix, we get

$$K_1 = \begin{bmatrix} 9 & -6 & 0 \\ -6 & 9 & -3 \\ 0 & -3 & 3 \end{bmatrix} \quad , \quad K_2 = \begin{bmatrix} 9 & 6 & 0 \\ 6 & 9 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

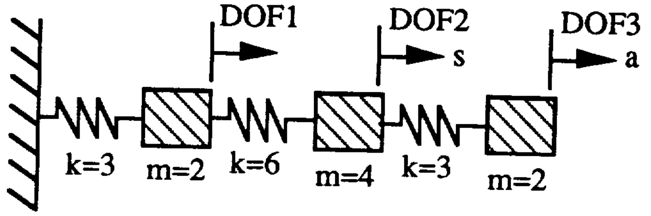

Figure 1: A three degrees-of-freedom mass-spring system.

in which K_1 is the correct stiffness matrix in physical coordinates. K_2 is not the correct stiffness matrix, but the (2,2), (2,3), (3,2), and (3,3) elements are correct. This is because that there is an actuator located at DOF3 and a sensor located at DOF2.

This example shows that with only one actuator and one sensor (non-colocated), the physical stiffness matrix of the three degrees-of-freedom mass-spring system can not be uniquely identified from the given modal model and the given physical locations of actuator and sensor. However, those elements in the stiffness matrix corresponding to the actuator and sensor locations are exactly and uniquely determined

Concluding Remarks

Identifiability of linear, time invariant systems from test data with a limited number of actuators and sensors is investigated in this paper. It is shown that for systems with unsymmetric system matrices, either the number of actuators or the number of sensors must be greater than or equal to the number of states so that the system matrices in physical coordinates can be uniquely identified. For systems with symmetric system matrices, the number of actuators plus the number of sensors must be greater than or equal to the number of states in order for the physical system matrices to be uniquely determined. For undamped structural dynamics systems with symmetric mass and stiffness matrices, the total number of actuators and sensors must be greater than or equal to the number of degrees-of-freedom, so that the physical mass and stiffness matrices can be identified from test data. A simple mass-spring system is used as an example to verify the conclusions.

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